Hölder spaces lecture notes

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1 Motivation

Over the next few lectures we want to establish an "regularity and compactness theory" for solutions to elliptic equations. By a regularity theory I mean theorem(s) stating that the regularity of a solution follows from the regularity of the coefficients, inhomogeneous term f, and other data. By a compactness theory I mean theorem(s) stating that given a sequence of solutions to elliptic equations and appropriate bounds, there exists a convergent subsequence. This will correspond to estimates called the Schauder estimates.

The simplest such theorem that one might imagine is that if u is a reasonable solution to an elliptic equation $Lu = f$ in the unit ball $B_1(0)$ and the coefficients and f are all continuous, then $u \in C^2(B_{1/2}(0))$ and

$$\|u\|_{C^2(B_{1/2}(0))} \equiv \sum_{|\alpha| \leq 2} \sup_{B_{1/2}(0)} |D^\alpha u| \leq C \left(\sup_{B_1(0)} |u| + \sup_{B_1(0)} |f| \right)$$

for some constant $C \in (0, \infty)$ depending only on n and L. Such a theorem is false!

Additionally, we know that given a sequence $\{u_j\}$ of C^2 functions (say solutions to elliptic equations) with $\sup_j \|u_j\|_{C^2(B_{1/2}(0))} < \infty$, then it is possible that $\{u_j\}$ converges to a function that is not in C^2. However, if we additionally showed that $\{u_j\}$ is equicontinuous, then we could apply Arzela-Ascoli to extract a subsequence of $\{u_j\}$ converging to a C^2 function i uniformly, and moreover the derivatives up to order two also converge uniformly.

Thus we will introduce a subset of $C^k(\Omega)$ known as Hölder spaces.

2 $C^{k,\mu}$ functions

Let Ω be an open set in $\mathbb{R}^n$, $k \geq 0$ be an integer, and $\mu \in (0, 1]$. Given a function $u : \Omega \to \mathbb{R}$, we let

$$[u]_{\mu,\Omega} = \sup_{x,y \in \Omega,\, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\mu}.$$

We can regard $[u]_{\mu,\Omega}$ as a measure of the modulus of continuity of u. In the special case that $\mu = 1$ and u is Lipschitz, $[u]_{1,\Omega}$ is the Lipschitz constant of u:

$$[u]_{1,\Omega} = \operatorname{Lip} u = \operatorname{ess\,sup}_\Omega |Du|.$$

Given functions $u, v : \Omega \to \mathbb{R}$,

$$[u+v]_{\mu;\Omega} \leq [u]_{\mu;\Omega} + [v]_{\mu;\Omega} \quad [uv]_{\mu;\Omega} \leq \sup_{\Omega} |u| \, [v]_{\mu;\Omega} + [u]_{\mu;\Omega} \sup_{\Omega} |v|.$$

Recall that $C^k(\Omega)$ is the space of all functions $u : \Omega \to \mathbb{R}$ such that $D^\alpha u$ exists and are continuous on Ω whenever $|\alpha| \leq k$. We define

$$C^{k,\mu}(\Omega) = \{u \in C^k(\Omega) : [D^\alpha u]_{\mu,\Omega'} < \infty \text{ whenever} |\alpha| \leq k \text{ and } \Omega' \subset\subset \Omega\},$$

where $\Omega' \subset\subset \Omega$ means that Ω' is an open subset of Ω whose closure $\overline{\Omega'}$ is compact. Note that in this definition of $C^{k,\mu}(\Omega)$ we do not say anything about the behavior of $u \in C^{k,\mu}(\Omega)$ at the boundary of Ω or at infinity, we only control the local modulus of continuity of $D^\alpha u$ in Ω for $|\alpha| \leq k$.

We let $C^k_c(\Omega)$ denote the set of $u \in C^k(\Omega)$ such that for some compact set $K \subset \Omega$, $u = 0$ on $\Omega \setminus K$. Similarly, we let $C^{k,\mu}_c(\Omega)$ denote the set of $u \in C^{k,\mu}(\Omega)$ such that for some compact set $K \subset \Omega$, $u = 0$ on $\Omega \setminus K$.

We define $C^k(\overline{\Omega})$ to be the set of $u \in C^k(\Omega)$ such that $D^\alpha u$ extends to continuous functions on $\overline{\Omega}$ whenever $|\alpha| \leq k$. As a slight abuse of notation, we will let $D^\alpha u$ denote the extension of $D^\alpha u$ to $\overline{\Omega}$. Note that if $u \in C^{k,\mu}(\overline{\Omega})$ and Ω is a C^1 domain, then for every $x \in \partial\Omega$, α with $|\alpha| \leq k-1$, and $\varepsilon > 0$ we can choose $\delta > 0$ such that the following holds true. There exists a C^1 diffeomorphism $\Psi : B_\rho(x) \to \mathbb{R}^n$ such that

$$\Psi(x) = 0, \quad D\Psi(x) = I_n, \quad |D\Psi(x) - I_n| \leq 1/2, \quad \Psi(B_\rho(x) \cap \Omega) \subseteq \{x \in B_1(0) : x_n > 0\}$$

(where I_m denotes the $m \times m$ identity matrix); for example, translate x to the origin and rotate so that

$$B_\rho(0) \cap \Omega = B_\rho(0) \cap \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > \psi(x')\}$$

for some C^1 function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\psi(0) = 0$, $D\psi(0) = 0$, $|D\psi|$ is small and let $\Psi(x', x_n) = (x', x_n - \psi(x'))$. Given $h \in B_\rho(0)$ with $x + h \in \overline{\Omega}$, let $\gamma(t) = \Psi^{-1}(t\Psi(x+h))$. By the fundamental theorem of calculus,

$$\begin{aligned}
|D^\alpha u(x+h) - D^\alpha u(x) - DD^\alpha u(x) \cdot h| &= \left| \int_0^1 DD^\alpha u(\gamma(t)) \cdot \gamma'(t) dt - \int_0^1 DD^\alpha u(x) \cdot \gamma'(t) dt \right| \\
&\leq \int_0^1 |DD^\alpha u(x+th) - DD^\alpha u(x)| |\gamma'(t)| dt \\
&\leq 4 |DD^\alpha u(x+th) - DD^\alpha u(x)| |h| < \varepsilon |h|,
\end{aligned}$$

where we use the fact that

$$|\gamma'(t)| = |D\Psi^{-1}(t\Psi(x+h)) \cdot \Psi(x+h)| \leq |D\Psi^{-1}(t\Psi(x+h))| |\Psi(x+h) - \Psi(x)| \leq 4|h|,$$

so $DD^\alpha u(x)$ is the derivative of $D^\alpha u$ at every $x \in \overline{\Omega}$ even when $x \in \partial\Omega$.

We let

$$C^{k,\mu}(\overline{\Omega}) = \{u \in C^k(\overline{\Omega}) : [D^\alpha u]_{\mu,\Omega} < \infty\}.$$

Given any open set Ω in $\mathbb{R}^n$ and integer $k \geq 0$, we can let

$$\|u\|_{C^k(\Omega)} = |u|_{k;\Omega} = \sum_{|\alpha| \leq k} \sup_{\Omega} |D^\alpha u|$$

for all $u \in C^k(\Omega)$. Additionally given $\mu \in (0,1]$, we can let

$$\|u\|_{C^{k,\mu}(\Omega)} = |u|_{k,\mu;\Omega} = \sum_{|\alpha| \leq k} |D^\alpha u|_{0;\Omega} + \sum_{|\alpha|=k} [D^\alpha u]_{\mu,\Omega}$$

for all $u \in C^k(\Omega)$. (Note that at the moment this is just notation and I say nothing about whether $\|u\|_{C^k(\Omega)}$ or $\|u\|_{C^{k,\mu}(\Omega)}$ are finite.) It is convenient to define a scale invariant "norms" by

$$\|u\|'_{C^k(\Omega)} = |u|'_{k;\Omega} = \sum_{|\alpha| \leq k} (d/2)^{|\alpha|} |D^\alpha u|_{0;\Omega},$$

$$\|u\|'_{C^{k,\mu}(\Omega)} = |u|'_{k,\mu;\Omega} = \sum_{|\alpha| \leq k} (d/2)^{|\alpha|} |D^\alpha u|_{0;\Omega} + \sum_{|\alpha|=k} (d/2)^{k+\mu} [D^\alpha u]_{\mu,\Omega},$$

where $d = \operatorname{diam} \Omega$ (for example, if $\Omega = B_R(x_0)$ is a ball then $d/2 = R$ is the radius of the ball). It is easily checked that if $u \in C^k(B_R(x_0))$ and $\tilde{u}(x) = u(x_0 + Rx)$, then

$$|u|'_{k,\mu;B_R(x_0)} = |\tilde{u}|_{k,\mu;B_1(0)}.$$

We say for $u_j, u \in C^k(\Omega)$ that $u_j \to u$ in $C^k(\Omega)$ if $D^\alpha u_j \to D^\alpha u$ uniformly in Ω' whenever $|\alpha| \leq k$ and $\Omega' \subset\subset \Omega$. Similarly we say for $u_j, u \in C^k(\overline{\Omega})$ that $u_j \to u$ in $C^k(\overline{\Omega})$ if $D^\alpha u_j \to D^\alpha u$ uniformly in Ω whenever $|\alpha| \leq k$.

Note that the spaces $C^{k,\mu}(\Omega)$ are nested in the sense that if $0 < \mu < \tau \leq 1$ then $C^{k,\tau}(\Omega) \subset C^{k,\mu}(\Omega)$ since if $u \in C^{k,\tau}(\Omega)$ and $\Omega' \subset\subset \Omega$ then

$$\begin{aligned}
[D^\alpha u]_{\mu,\Omega'} &= \sup_{x,y \in \Omega', x \neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\mu} \\
&= \sup_{x,y \in \Omega', x \neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\tau} \cdot |x-y|^{\tau-\mu} \\
&\leq \operatorname{diam}(\Omega')^{\tau-\mu} [D^\alpha u]_{\tau,\Omega'} < \infty.
\end{aligned}$$

Similarly, if Ω is a bounded $C^{k,\tau}$ domain and $0 < \mu < \tau \leq 1$ then $C^{k,\mu}(\overline{\Omega}) \subset C^{k,\tau}(\overline{\Omega})$.

3 Compactness theorems

As was claimed previously, Arzela-Ascoli yields compactness theorems for Hölder spaces:

Theorem 1. *Let Ω be an open set in $\mathbb{R}^n$, $k \geq 0$, and $\mu \in (0,1]$. Given a sequence of $u_j \in C^{k,\mu}(\Omega)$ such that*

$$\sup_j |u_j|_{k,\mu;\Omega'} < \infty \text{ for all } \Omega' \subset\subset \Omega$$

there exists a subsequence $\{u_{j'}\}$ of $\{u_j\}$ and a function $u \in C^{k,\mu}(\Omega)$ such that $u_j \to u$ in $C^k(\Omega)$. (Note that we do not claim that $u_j \to u$ in $C^{k,\mu}(\Omega)$, i.e. $|u_j - u|_{k,\mu;\Omega'} \to 0$ for all $\Omega \subset\subset \Omega$.)

Theorem 2. *Let Ω be a bounded, open, C^1 domain in $\mathbb{R}^n$, $k \geq 0$, and $\mu \in (0,1]$. Given a sequence of $u_j \in C^{k,\mu}(\overline{\Omega})$ such that*

$$\sup_j |u_j|_{k,\mu;\Omega} < \infty \tag{1}$$

there exists a subsequence $\{u_{j'}\}$ of $\{u_j\}$ and a function $u \in C^{k,\mu}(\overline{\Omega})$ such that $u_j \to u$ in $C^k(\overline{\Omega})$. (Note that we do not claim that $u_j \to u$ in $C^{k,\mu}(\overline{\Omega})$, i.e. $|u_j - u|_{k,\mu;\Omega} \to 0$.)

The proofs are similar so let's prove Theorem 2.

Proof of Theorem 2. Let

$$\Lambda = \sup_j |u_j|_{k,\mu;\Omega}.$$

By (1), for $|\alpha| \leq k$, the sequence $\{D^\alpha u_j\}$ is pointwise uniformly bounded on $\overline{\Omega}$ as $\sup_\Omega |D^\alpha u_j| \leq \Lambda < \infty$. For $|\alpha| < k$, $\{D^\alpha u_j\}$ is also equicontinuous on $\overline{\Omega}$ since $[D^\alpha u_j]_{1;\Omega} = \sup_\Omega |DD^\alpha u_j| \leq \Lambda < \infty$. To see this, observe that since Ω is a C^1 domain, for every $y \in \partial\Omega$ there exists a $\rho_y > 0$ and C^1 diffeomorphism $\Psi_y : B_{\rho_y}(y) \to \mathbb{R}^n$ such that

$$\Psi_y(y) = 0, \quad D\Psi_y(y) = I_n, \quad |D\Psi_y(x) - I_n| \leq 1/2, \quad \Psi(B_{\rho_y}(y) \cap \Omega) \subseteq \{x \in B_1(0) : x_n > 0\}.$$

$\{D^\alpha u_j \circ \Psi_y^{-1}\}$ is equicontinuous on $B_{\rho_y/2}(y) \cap \{x \in B_1(0) : x_n > 0\}$ since given $\varepsilon > 0$ there exists $\delta = \delta(y) > 0$ independent of j such that

$$|(D^\alpha u_j \circ \Psi_y^{-1})(z) - (D^\alpha u_j \circ \Psi_y^{-1})(z')| \leq \sup |DD^\alpha u_j| \sup |D\Psi_y^{-1}||z - z'| \leq 2\Lambda\delta < \varepsilon$$

for all $z, z' \in B_{\rho_y/2}(y) \cap \{x \in B_1(0) : x_n > 0\}$ with $|z - z'| < \delta$ and for all j. Hence $\{D^\alpha u_j\}$ is equicontinuous on $B_{\rho_y/4}(y) \cap \Omega$. Cover $\partial\Omega$ by a finite collection of balls $\{B_{\rho_{y_k}/8}(y_k)\}$ where $y_k \in \partial\Omega$ and let $\rho = \min_k \rho_{y_k}$. For every $\varepsilon > 0$ there exists $\delta \in (0, \rho/16)$ independent of j such that

$$|D^\alpha u_j(z) - D^\alpha u_j(z')| \leq \sup |DD^\alpha u_j||z - z'| \leq \Lambda\delta < \varepsilon$$

for all $z, z' \in \Omega$ with $\operatorname{dist}(z, \partial\Omega) > \rho/16$ with $|z - z'| < \delta$ and for all j. Combining this with $\{D^\alpha u_j\}$ being equicontinuous on each $B_{\rho_{y_k}/4}(y_k) \cap \Omega$, $\{D^\alpha u_j\}$ is equicontinuous on Ω. For $|\alpha| = k$, the sequence $\{D^\alpha u_j\}$ is equicontinuous since $[D^\alpha u_j]_{\mu;\Omega} \leq \Lambda < \infty$ and thus given $\varepsilon > 0$ we can choose $\delta > 0$ independent of j that

$$|D^\alpha u_j(z) - D^\alpha u_j(z')| \leq \Lambda\delta^\mu < \varepsilon$$

for all $z, z' \in \Omega$ with $|z - z'| < \delta$ and j. Therefore, for $|\alpha| \leq k$, by Arzela-Ascoli we can pass to a subsequence of $\{D^\alpha u_j\}$ that converges uniformly to some continuous function $v_\alpha : \Omega \to \mathbb{R}$.

For $k \geq 1$ we need to check that $v_\alpha = D^\alpha u$ for all α. By the fundamental theorem of calculus and (1), for every α with $|\alpha| = k - 1$, $x \in \Omega$, and $\varepsilon > 0$ we can choose $\delta > 0$ independent of j such that $B_\delta(x) \subset\subset \Omega$ and

$$\begin{aligned}
\left| D^\alpha u_j(x + h) - D^\alpha u_j(x) - \sum_{i=1}^n D_i D^\alpha u_j(x) h_i \right| &= \left| \int_0^1 DD^\alpha u_j(x + th) \cdot h dt - DD^\alpha u_j(x) \cdot h \right| \\
&\leq \int_0^1 |DD^\alpha u_j(x + th) - DD^\alpha u_j(x)||h| dt \\
&\leq [DD^\alpha u_j]_{\mu;\Omega} |h|^{1+\mu} \\
&\leq \Lambda |h|^{1+\mu} < \varepsilon |h|
\end{aligned}$$

for all h with $|h| < \delta$ and j. Similarly when α with $|\alpha| \leq k - 2$, $x \in \Omega$, and $\varepsilon > 0$ we can choose $\delta > 0$ independent of j such that $B_\delta(x) \subset\subset \Omega$ and

$$\left| D^\alpha u_j(x + h) - D^\alpha u_j(x) - \sum_{i=1}^n D_i D^\alpha u_j(x) h_i \right| \leq \sup_\Omega |D^2 D^\alpha u_j||h|^2 \leq \Lambda |h|^2 < \varepsilon |h|$$

for all h with $|h| < \delta$ and j. Letting $j \to \infty$,

$$\left| v_\alpha(x+h) - v_\alpha(x) - \sum_{i=1}^{n} v_{\alpha+e_i}(x) h_i \right| < \varepsilon |h|,$$

where $e_1, e_2, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$ and thus $\alpha + e_i$ denotes the multi-index in which we replace α_i by $\alpha_i + 1$. Therefore $D_i v_\alpha = v_{\alpha+e_i}$ for all α with $|\alpha| \leq k-1$ and $i = 1, 2, \ldots, n$.

Finally we need to check that $[D^\alpha u]_{\mu;\Omega} < \infty$ for $|\alpha| = k$. By (1),

$$|D^\alpha u_j(x) - D^\alpha u_j(y)| \leq \Lambda |x-y|^\mu$$

for all $x, y \in \Omega$ and j. Letting $j \to \infty$, using merely the fact that $D^\alpha u_j \to D^\alpha u$ uniformly in Ω,

$$|D^\alpha u(x) - D^\alpha u(y)| \leq \Lambda |x-y|^\mu$$

for all $x, y \in \Omega$. □

4 Interpolation

Theorem 3. *Let k and l be integers such that $1 \leq k \leq l$ and $\mu \in (0,1]$. For every $\varepsilon > 0$, for every $u \in C^{l,\mu}(\overline{B_R(0)})$,*

$$R^k |D^k u|_{0;B_R(0)} \leq C |u|_{0;B_R(0)} + \varepsilon R^{l+\mu} [D^l u]_{\mu;B_R(0)} \tag{2}$$

for some constant $C = C(\varepsilon, k, l, \mu) \in (0, \infty)$.

Proof. Obviously we may rescale as to assume that $R = 1$. The rest of the proof will be an exercise. □

5 Extension theorems

Define $\mathbb{R}^n_+ = \{x : x_n > 0\}$ and $\mathbb{R}^n_- = \{x : x_n < 0\}$. For $R > 0$, let $B^+_R = B_R(0) \cap \mathbb{R}^n_+$ and $B^-_R = B_R(0) \cap \mathbb{R}^n_-$.

Theorem 4 (Extension Lemma)**.** *Let $k \geq 1$ be an integer and $\mu \in (0,1]$. Let Ω be a bounded $C^{k,\mu}$ domain and let Ω' be an open set containing $\overline{\Omega}$. Then every function $u \in C^{k,\mu}(\Omega)$ has an extension $\bar{u} \in C^{k,\mu}_c(\Omega')$ such that $\bar{u} = u$ on Ω and $|\bar{u}|_{k,\mu,\Omega'} \leq C |u|_{k,\mu,\Omega}$ for some constant $C = C(n, k, \mu, \Omega, \Omega') \in (0, \infty)$ independent of u.*

Proof. Since Ω is a $C^{k,\mu}$ domain, for every $\xi \in \partial\Omega$, there is a $\delta_\xi > 0$ and C^k diffeomorphism $\Psi_\xi : B_{\delta_\xi}(\xi) \to \Psi_\xi(B_{\delta_\xi}(\xi)) \subseteq \mathbb{R}^n$ such that

$$\begin{aligned} &\Psi_\xi(\Omega \cap B_{\delta_\xi}(\xi)) \subseteq \mathbb{R}^n_+, \\ &\Psi_\xi(\partial\Omega \cap B_{\delta_\xi}(\xi)) \subseteq \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}. \end{aligned}$$

We may assume $B_{\delta(\xi)}(\xi) \subset\subset \Omega'$, $\Psi_\xi(\xi) = 0$, and $B^+_1(0) \subseteq \Psi_\xi(B_{\delta_\xi}(\xi)$.

Let $\tilde{u}_\xi = u \circ \Psi_\xi^{-1}$ on $B_1^+(0)$. We extend $\tilde{u}_\xi$ to all of $B_1(0)$ by letting

$$\tilde{u}_\xi(x', x_n) = \sum_{j=1}^{k+1} c_j \tilde{u}_\xi(x', -x_n/j)$$

for $x = (x_1, x_2, \ldots, x_n) \in B_1(0)$ with $x_n < 0$, where $x' = (x_1, x_2, \ldots, x_{n-1})$ and

$$\sum_{j=1}^{k+1} c_j(-1/j)^m = 1 \text{ for } m = 0, \ldots, k.$$

The c_j are the unique solution to a linear system with a Vandermonde matrix. We compute for all α with $|\alpha| \leq k$ and all $x = (x', x_n) \in B_1(0)$ with $x_n \leq 0$ that

$$D^\alpha \tilde{u}_\xi(x', x_n) = \sum_{j=1}^{k+1} c_j(-1/j)^{\alpha_n} D^\alpha \tilde{u}_\xi(x', -x_n/j)$$

and in particular when $x_n = 0$,

$$D^\alpha \tilde{u}_\xi(x', 0) = \sum_{j=1}^{k+1} c_j(-1/j)^{\alpha_n} D^\alpha \tilde{u}_\xi(x', 0) = D^\alpha \tilde{u}_\xi(x', 0),$$

so $\tilde{u}_\xi \in C^k(B_1(0))$. (Note that all this really shows is that $D^\alpha \tilde{u}_\xi$ is continuous across $B_1(0) \cap \{x_n = 0\}$, not quite that $\tilde{u}_\xi$ is continuously differentiable up to order k at points in $B_1(0)$. However, $\tilde{u}_\xi \in C^{k,\mu}(B_1^+(0))$ and the reflection of $\tilde{u}_\xi$ across $\{x_n = 0\}$ is consequently in $C^{k,\mu}(B_1^-(0))$, so it follows from our discussion in Section 2 above that $\tilde{u}_\xi$ is continuously differentiable up to order k on $B_1(0)$.) Also, for every α with $|\alpha| = k$ and every $x = (x', x_n)$ and $y = (y', y_n)$ in $B_1^-(0)$,

$$\begin{aligned}
|D^\alpha u(x', x_n) - D^\alpha u(y', y_n)| &\leq \sum_{j=1}^{k+1} c_j (1/j)^{\alpha_n} |D^\alpha \tilde{u}_\xi(x', -x_n/j) - D^\alpha \tilde{u}_\xi(y', -y_n/j)| \\
&\leq \sum_{j=1}^{k+1} c_j (1/j)^{\alpha_n} [D^\alpha \tilde{u}_\xi]_{\mu; B_1^+(0)} (|x' - y'|^\mu + |x_n - y_n|^\mu / j) \\
&\leq \sum_{j=1}^{k+1} c_j [D^\alpha \tilde{u}_\xi]_{\mu; B_1^+(0)} |x - y|^\mu,
\end{aligned}$$

so $[D^\alpha \tilde{u}_\xi]_{k,\mu,B_1^-(0)} \leq C[D^\alpha \tilde{u}_\xi]_{k,\mu,B_1^+(0)}$ for $C = C(n,k,\mu) \in (0,\infty)$. It readily follows that $\tilde{u}_\xi \in C^{k,\mu}(B_1(0))$ and $|\tilde{u}_\xi|_{k,\mu;B_1(0)} \leq C|\tilde{u}_\xi|_{k,\mu,B_1^+(0)}$ for $C = C(n,k,\mu) \in (0,\infty)$. Therefore, $\tilde{u}_\xi \circ \Psi_\xi$ is an extension of u to $\Psi_\xi^{-1}(B_1(0))$ with $|\tilde{u}_\xi \circ \Psi_\xi|_{k,\mu;\Psi_\xi^{-1}(B_1(0))} \leq C|u|_{k,\mu;\Omega}$.

Find a finite subcover $\{V_i = \Psi_{\xi_i}^{-1}(B_1(0)) : i = 1, \ldots, N\}$ of $\partial\Omega$, where $\xi_1, \ldots, \xi_N \in \partial\Omega$. Then $\{V_i : i = 1, 2, \ldots, N\} \cup \{\Omega\}$ covers $\overline{\Omega}$. Find a partition of unity χ_i subordinate to $\{V_i\} \cup \{\Omega\}$; that is, find $\chi_i \in C_c^\infty(\Omega')$ such that $\chi_0 = 0$ on $\Omega' \setminus \Omega$, $\chi_i = 0$ on $\Omega \setminus V_i$ for $i = 1, 2, \ldots, N$, and

$$\sum_{i=1}^{\infty} \chi_i = 1 \text{ on } \overline{\Omega}.$$

Let $\bar{u}_i$ denote the extension of u to V_i constructed in the preview paragraph. Define

$$\bar{u} = \chi_0 u + \sum_{i=1}^{\infty} \chi_i \bar{u}_i \text{ on } \Omega'.$$

Obviously,

$$\bar{u} = \sum_{i=0}^{\infty} \chi_i u = u \text{ on } \Omega.$$

and $|\bar{u}|_{k,\mu,\Omega'} \leq C|u|_{k,\mu,\Omega}$ for $C = C(n,k,\mu,\Omega,\Omega') > 0$. □

Note that what the proof of the Extension Theorem shows is that there exists a bounded linear extension operator

$$E : C^{k,\mu}(\overline{\Omega}) \to C_c^{k,\mu}(\Omega')$$

with $Eu = u$ on $\overline{\Omega}$ for every integer $k \geq 1$, $\mu \in (0,1]$, bounded $C^{k,\mu}$ domain Ω, and open set Ω' containing $\overline{\Omega}$. Thus if $R : C_c^{k,\mu}(\Omega') \to C^{k,\mu}(\overline{\Omega})$ is the restriction operator $Ru = u|_{\Omega}$, then $R \circ E$ is the identity map.

We also have the following extension theorem for $\varphi \in C^{k,\mu}(\partial\Omega)$ in the case that Ω is a $C^{k,\mu}$ domain. Recall that since Ω is a $C^{k,\mu}$ domain, $\partial\Omega$ is a $C^{k,\mu}$, $(n-1)$-dimensional submanifold since for every $\xi \in \partial\Omega$, there is a $\delta_\xi > 0$ and C^k diffeomorphism $\Psi_\xi : B_{\delta_\xi}(\xi) \to \Psi_\xi(B_{\delta_\xi}(\xi)) \subseteq \mathbb{R}^n$ such that

$$\Psi_\xi(\partial\Omega \cap B_{\delta_\xi}(\xi)) \subseteq \mathbb{R}^{n-1} \times \{0\}.$$

Thus by $\varphi \in C^{k,\mu}(\partial\Omega)$ we mean that $\varphi \circ (\Psi_\xi|_{\mathbb{R}^{n-1}\times\{0\}})^{-1}$ is in $C^{k,\mu}$ for all $\xi \in \partial\Omega$ (note that the choice of Ψ_ξ is irrelevant by the chain rule).

Theorem 5. *Let $k \geq 1$ be an integer and $\mu \in (0,1]$. Let Ω be a bounded $C^{k,\mu}$ domain and let Ω' be an open set containing $\overline{\Omega}$. Then every function $\varphi \in C^{k,\mu}(\partial\Omega)$ has an extension $\bar{\varphi} \in C_c^{k,\mu}(\Omega')$ such that $\bar{\varphi} = \varphi$ on $\partial\Omega$ and $|\bar{\varphi}|_{k,\mu,\Omega'} \leq C|\varphi|_{k,\mu,\partial\Omega}$ for some constant $C = C(n,k,\mu,\Omega,\Omega') \in (0,\infty)$ independent of φ.*

Proof. The proof is similar to the proof of the Extension Theorem. Let $\xi \in \partial\Omega$ and Ψ_ξ be as in the proof of the Extension Theorem. We want to extend $\tilde{\varphi}_\xi = \varphi \circ \Psi_\xi^{-1}$ from $B_1^{n-1}(0) \times \{0\}$ to $B_1^{n-1}(0) \times \mathbb{R}$. We do so by letting

$$\tilde{\varphi}_\xi(x', x_n) = \tilde{\varphi}_\xi(x', 0)$$

for all $x' \in B_1^{n-1}(0)$ and $x_n \in \mathbb{R}$. It is easily checked that $\tilde{\varphi}_\xi \in C^{k,\mu}(B_1^{n-1}(0) \times \mathbb{R})$ and $|\tilde{\varphi}_\xi|_{k,\mu,B_1^{n-1}(0)\times\mathbb{R}} \leq C|\tilde{\varphi}_\xi|_{k,\mu,B_1^{n-1}(0)\times\{0\}}$ for $C = C(n,k,\mu) \in (0,\infty)$. Thus $\tilde{\varphi}_\xi \circ \Psi_\xi$ is an extension of φ to $\Psi_\xi^{-1}(B_1(0))$ with $|\tilde{\varphi}_\xi \circ \Psi_\xi^{-1}|_{k,\mu;\Psi_\xi^{-1}(B_1(0))} \leq C|\varphi|_{k,\mu;\partial\Omega}$. Let $\{V_i = \Psi_{\xi_i}^{-1}(B_1(0)) : i = 1,\ldots,N\}$ be a finite cover of $\partial\Omega$, where $\xi_1,\ldots,\xi_N \in \partial\Omega$. Let χ_i be the partition of unity subordinate to $\{V_i\} \cup \{\Omega\}$. Define

$$\bar{\varphi} = \chi_0 \varphi + \sum_{i=1}^{\infty} \chi_i \bar{\varphi}_i \text{ on } \Omega'.$$

Obviously, $\bar{\varphi} = \varphi$ on $\partial\Omega$ and $|\bar{\varphi}|_{k,\mu,\Omega'} \leq C|\varphi|_{k,\mu,\partial\Omega}$ for $C = C(n,k,\mu,\Omega,\Omega') > 0$. □

References: Gilbarg and Trudinger, Section 4.1.